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Sine-Gordon Soliton on a Cnoidal Wave Background

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ABSTRACT

The method of Darboux transformation, which is applied on cnoidal wave solutions of the sine-Gordon equation, gives solitons moving on a cnoidal wave background. Interesting characteristics of the solution, i.e., the velocity of solitons and the shift of crests of cnoidal waves along a soliton, are calculated. Solutions are classified into three types (Type-1A, Type-1B, Type-2) according to their apparent distinct properties.

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1 Introduction

The sine-Gordon equation is important for a number of physical and mathematical systems where topological solitons are present. In the case of physical applications, it is a canonical model describing charge density waves [1], fluxon dynamics in Josephson junctions [2], and DNA promoter dynamics [3], for example. In the case of mathematical topics, we can mention the embedding of a surface with constant negative curvature in the 3-dimensional space [4]. In theoretical physics, it was studied as a model system of hadrons with the idea that extended particles in quantum field theory can be associated with soliton solutions of integrable equations [5]. The quantum sine-Gordon model is also developed and the exact S-matrix of the solitons was derived by using the bootstrap methods [6].

The important property of the sine-Gordon equation in these studies is the existence of soliton solutions. The simple functional form of N-soliton has led to a wide range of scientific applications. Considering the potential applicability of the soliton solutions, it is desired to have a more generalized form of soliton solutions, having more parameters and characteristics. One possible scheme in this direction is constructing soliton solutions lying on a background field. It would give more freedom in controlling and characterizing solitons by using proper backgrounds. Indeed, the sine-Gordon equation, as an integrable equation, has solutions of superposed states "soliton + cnoidal wave". Recently, this type of solutions have attracted new interest in nonlinear optics, described by the nonlinear Schrödinger equation (NLSE), due to the possibility of dynamically reconfigurable photonic structures [7, 8, 9].

In fact, the "soliton + cnoidal wave" solution was obtained from the general quasi-periodic solutions of N-phase theta functions [10, 11], by taking degenerate limit of the 2-phase solution [12, 13]. To apply solutions from the N-phase theta functions to real physical problems, the authors in [12, 13] solve the so-called "effectivization" problem, which is related to extracting physical solutions by taking proper initial conditions [12, 13, 14, 15]. However we find that above solutions, though explicit, are inconvenient to analyze their properties, as the parameters in the solutions are related in a complicated form. In fact, the mathematical and computational complexities of computing N-phase theta functions have hindered the use of these solutions in applications except some clever calculations like in [16, 17].

In this paper, we employ a simple, but powerful soliton finding technique [8, 9] based on the Darboux transformation (DT) to find solutions of "soliton + cnoidal wave". Compared to the results from the N-phase theta functions, the form of solution, especially the parameters, are simple and convenient to analyze various

properties of the solutions. Explicit real solutions are described by a parameter u without any further constraints and it is directly related to the characteristics of solutions. This DT technique gives a clear classification scheme of the solutions (Type-1A, Type-1B, Type-2) according to their apparent distinct properties like the modulating form of cnoidal waves and the velocities of solitons. The velocity of solitons are expressed in a form of relativistic addition, which is a unique feature of the "soliton + cnoidal wave" solution of the *relativistic* sine-Gordon equation and is essentially different from that of the *nonrelativistic* NLSE and KdV equation. As the sine-Gordon theory has various physical applications, our solution could yield a new interest on controlling the soliton using background fields.

2 sine-Gordon equation

2.1 Cnoidal wave solutions

The sine-Gordon equation is

$$\partial\bar{\partial}\phi = 2\beta\sin 2\phi, \tag{1}$$

where $\partial = \frac{\partial}{\partial z}, \bar{\partial} = \frac{\partial}{\partial \bar{z}}$, and $\bar{z} \equiv t - x$ and $z \equiv t + x$ represent the lightcone coordinates, while x, t are ordinary space-time coordinates. It has two types of cnoidal wave solutions;

$$\begin{aligned} \phi_c^{(1)}(z,\bar{z}) &= \sin^{-1}(\mathrm{sn}(\chi,k)),\ \partial\phi_c^{(1)} = \frac{2}{k}\sqrt{\frac{\beta}{V}}\mathrm{dn}(\chi,k), \qquad \mathrm{Type}-1 \\ \phi_c^{(2)}(z,\bar{z}) &= \sin^{-1}(k\ \mathrm{sn}(k\chi,k)),\ \partial\phi_c^{(2)} = 2k\sqrt{\frac{\beta}{V}}\mathrm{cn}(k\chi,k), \qquad \mathrm{Type}-2 \end{aligned} \tag{2}$$

where

$$\chi = \frac{2}{k}\sqrt{\frac{\beta}{V}}(z - V\bar{z}) = \frac{4}{k}\sqrt{\beta}\frac{x - v_c t}{\sqrt{1 - v_c^2}}, \tag{3}$$

and sn, dn are the standard Jacobi elliptic functions. V is related to the velocity v_c of the cnoidal wave as $V = (1 + v_c)/(1 - v_c)$ and $k \in (0, 1)$ is the modulus of the Jacobi function. Here we consider the case of $V > 0, |v_c| < 1$ and $\beta > 0$. The $V < 0, |v_c| > 1$ case, corresponding to the rescaled inverted pendulum [17], can be obtained by considering the $x \leftrightarrow t, \beta \leftrightarrow -\beta$ symmetry of the sine-Gordon theory. As far as elliptic functions are involved we employ terminology and notation of Ref.

[18] without further explanations. $\phi_c^{(1)}$ of the Type-1 solution is a monotonically increasing or decreasing function of χ, while $\phi_c^{(2)}$ of the Type-2 is an oscillating function of χ.

2.2 Lax pair

To obtain a superposed "soliton + cnoidal wave" solution using the DT method [19, 20, 21], we need to find a solution of the following linear equations associated to the sine-Gordon equation (Lax pair),

$$
\begin{aligned}
(\partial + i\beta\lambda)s_1 + i(\partial\phi_c^{(i)})s_2 = 0, \quad (\partial - i\beta\lambda)s_2 + i(\partial\phi_c^{(i)})s_1 = 0,\\
(\bar{\partial} - \frac{i}{\lambda}\cos(2\phi_c^{(i)}))s_1 + \frac{1}{\lambda}\sin(2\phi_c^{(i)})s_2 = 0,\\
(\bar{\partial} + \frac{i}{\lambda}\cos(2\phi_c^{(i)}))s_2 - \frac{1}{\lambda}\sin(2\phi_c^{(i)})s_1 = 0,
\end{aligned}
\tag{4}
$$

where λ is an pure imaginary number and $i = 1, 2$ represents two types of cnoidal solutions in Eq. (2). We note that the compatibility between the equations (4) requires $\phi_c^{(i)}$ satisfies the Eq. (1).

2.3 Sym's solution

The solution of the linear equations (4) for the cnoidal wave ($\phi_c^{(i)}, i = 1, 2$) can be obtained by a slight modification of Sym's solution [22]. Sym first introduced this type of solution in a context of vortex motion in hydrodynamics. It was then applied to NLSE-related problems in Ref. [23, 8, 9]. A more detailed proof of Sym's solution (in a slightly different notation) is given in the Appendix of Ref. [9]. The Sym's solution in the case of sine-Gordon equation for the Type-1 cnoidal solution, $\phi_c^{(1)}$, is (to be called as the Type-1A solution)

$$
\begin{aligned}
s_1 &= \{ae^{M}\Theta_s(\chi - u) + ib\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)\mathrm{dn}(\chi + u, k)}e^{-M}\Theta_s(\chi + u)\}/\Theta_s(\chi),\\
s_2 &= \{-ia\frac{\mathrm{sn}(u,k)}{\mathrm{cn}(u,k)\mathrm{dn}(\chi - u, k)}e^{M}\Theta_s(\chi - u) + be^{-M}\Theta_s(\chi + u)\}/\Theta_s(\chi),
\end{aligned}
\tag{5}
$$

where a, b are arbitrary real parameters. Here,

$$
M = \sqrt{V\beta}\frac{\mathrm{cn}(u,k)^4 + (k^2-1)\mathrm{sn}(u,k)^4}{k\mathrm{sn}(u,k)\ \mathrm{cn}(u,k)\mathrm{dn}(u,k)}\bar{z} + \left(\frac{\Theta_s'(u)}{\Theta_s(u)} + \frac{\mathrm{dn}(u,k)(1-2\mathrm{sn}(u,k)^2)}{2\mathrm{sn}(u,k)\ \mathrm{cn}(u,k)}\right)\chi,
\tag{6}
$$

and

$$\Theta_s(u) = \theta_0(\frac{\pi u}{2K}) = 1 + 2\sum(-)^n q^{n^2}\cos(\frac{n\pi u}{K}), \tag{7}$$

with $q = \exp(-\pi K'/K)$ and Jacobi theta function θ_0. The parameter u is related to λ as

$$\lambda = \frac{i}{\sqrt{V\beta}}\frac{\mathrm{dn}(u,k)}{k\ \mathrm{cn}(u,k)\ \mathrm{sn}(u,k)}. \tag{8}$$

3 Type-1A soliton solution on a dn-type background

3.1 soliton solution

A new solution ϕ_{c-s}, describing a soliton moving on a cnoidal wave $\phi_c^{(1)}$, can be constructed using the DT method [19, 20, 21] as following;

$$\partial\phi_{c-s}(z,\bar{z}) = \partial\phi_c^{(1)}(z,\bar{z}) - 4\beta\lambda\frac{s_1 s_2}{s_1^2 - s_2^2},$$
$$\sin(2\phi_{c-s}) = 4i\frac{s_1 s_2(s_1^2+s_2^2)}{(s_1^2-s_2^2)^2}\cos(2\phi_c^{(1)}) - \frac{s_1^4+6s_1^2 s_2^2+s_2^2}{(s_1^2-s_2^2)^2}\sin(2\phi_c^{(1)}). \tag{9}$$

Using the fact that $s_i, i = 1, 2$ in Eq. (5) satisfy the associated linear equations (4), it can be proved that ϕ_{c-s} in Eq. (9) satisfies the sine-Gordon equation (1). It can be seen that the reality of ϕ_{c-s} requires u, as well as a, b, are real in the Type-1A solution.

Figure 1(a) plots $\partial\phi_{c-s}$ as a function of time t and space x. It is obtained by using Eqs. (9), (2), (5) and (6). It shows a characteristic soliton of sine-Gordon equation lying on a cnoidal wave background. The parameters used are $V = 1, k = 0.8, u = 0.7, \beta = 1, a = b = 1$. Figure 1(b) shows a time sliced view of the soliton in Figure 1(a) at $t = 0$. These figures are drawn using MATHEMATICA, which is also used to check that the solution in Eq. (9) indeed satisfies the equation of motion (1).

3.2 Properties of the solution

3.2.1 Shift of cnoidal crests

In figure 1, we can see that ϕ_{c-s} becomes a cnoidal wave when we move away from the soliton. In fact, as $M \to \infty$, $s_2 \to -i\mathrm{sn}(u,k)\mathrm{dn}(\chi-u,k)s_1/\mathrm{cn}(u,k)$ and $\partial\phi_{c-s} \to$

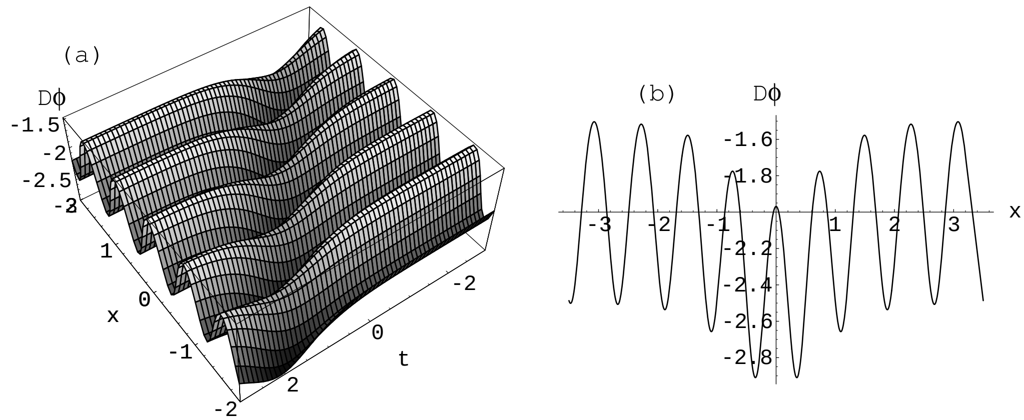

Figure 1: $\partial\phi_{c-s}$ showing a sine-Gordon soliton residing on a cnoidal wave background (Type-1A solution); (a) 3-dimensional plot, (b) time-sliced view at $t = 0$. The parameters are $V = 1, k = 0.8, u = 0.7, \beta = 1, a = b = 1$.

$-2\sqrt{\beta/V}\mathrm{dn}(\chi-2u,k)/k$, which is a cnoidal wave. Similarly, in the region $M \to -\infty$, $s_2 \to -i\mathrm{cn}(u,k)s_1/(\mathrm{dn}(\chi+u,k)\mathrm{sn}(u,k))$ and $\partial\phi_{c-s} \to -2\sqrt{\beta/V}\mathrm{dn}(\chi+2u,k)/k$. This calculation shows that crests of the cnoidal wave are shifted by $4u$ across the soliton, which can be seen in Fig. 1. Explicitly, the shift of crests for parameters of Fig. 1 is $4u = 2.8$ in χ and $ku = 0.56$ ($V = 1, v_0 = 0, \beta = 1$) in x, use Eq. (3).

3.2.2 velocity of the soliton

The soliton in Fig. 1 moves along a line described by $M = 0$, which can be casted in a form $0 = z + \Delta\bar{z}$ with

$$\Delta = -V + V\frac{\mathrm{cn}(u,k)^4 + (k^2-1)\mathrm{sn}(u,k)^4}{\mathrm{sn}(u,k)\ \mathrm{cn}(u,k)\mathrm{dn}(u,k)} \Big/ \left(2\frac{\Theta_s'(u)}{\Theta_s(u)} + \frac{\mathrm{dn}(u,k)(1-2\mathrm{sn}(u,k)^2)}{\mathrm{sn}(u,k)\ \mathrm{cn}(u,k)}\right). \tag{10}$$

Using the space-time coordinates t, x instead of the lightcone coordinates $z, \bar{z}$, it is written as $0 = z + \Delta\bar{z} = 2(x - vt)/(1 - v)$ where $v = (\Delta+1)/(\Delta-1)$. It shows v is the velocity of the soliton in the ordinary space-time coordinates. We can express the soliton velocity v in terms of the v_c ($V = (1 + v_c)/(1 - v_c)$) of the cnoidal wave

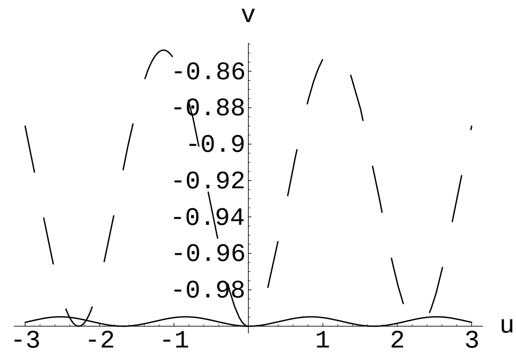

Figure 2: Intrinsic velocity v_0 versus u, Type-1A sine-Gordon soliton for $k = 0.5$ (solid line) and $k = 0.9$ (dashed line).

as $v = (v_c + v_0)/(1 + v_c v_0)$ where

$$v_0 = v(v_c = 0) = \frac{k^2 \mathrm{sn}^2 u \ \mathrm{cn}^2 u + \mathrm{dn}^2 u \ (1 - 2\mathrm{sn}^2 u)}{-4\mathrm{sn}u \ \mathrm{cn}u \ \mathrm{dn}u \ \Theta_s'(u)/\Theta_s(u) + k^2 \mathrm{sn}^2 u \ \mathrm{cn}^2 u - \mathrm{dn}^2 u \ (1 - 2\mathrm{sn}^2 u)}, \tag{11}$$

where $\mathrm{sn}u = \mathrm{sn}(u, k)$, etc. It shows that v_0 is the intrinsic velocity of the soliton observed in the moving frame of the cnoidal wave, which depends on the parameters u and k. The soliton velocity v is given by the relativistic addition of two velocities, v_c (velocity of the cnoidal wave) and v_0 (intrinsic velocity of the soliton), where v_c does not depend on k or u. Figure 2 shows the intrinsic velocity v_0 in u for a soliton on cnoidal wave backgrounds of $k = 0.5$ (solid line), and $k = 0.9$ (dotted line). Note that v_0 is a periodic function in u with the periodicity K. ($K(k = 0.5) = 1.69, K(k = 0.9) = 2.28$) It shows that the intrinsic velocity $v_0 = -1$ at $u = nK$ for integer n. Using Eq. (11) and the relation

$$E(\sin^{-1}(\mathrm{sn}u)) = \frac{\Theta_s'(u)}{\Theta_s(u)} + \frac{E}{K}u, \tag{12}$$

we can find that the velocity $v_0 = -k'K/E$ at $u = K/2$. Especially, this value is -0.995 for $k = 0.5$, -0.848 for $k = 0.9$, -1 for $k = 0$, and 0 for $k = 1$. Thus a soliton can be stopped only when it resides on a cnoidal wave with $k = 1$. On a $k = 0$ background, the soliton moves with a velocity $v_0 = -1$. Note that there is a symmetry, $t \to -t$, in the "soliton + cnoidal wave" solution in Eq. (9). This symmetry gives a solution having a soliton velocity $-v_0$.

4 Type-1B soliton solution on a dn-type background

4.1 soliton solution

The reality requirement of ϕ_{c-s} was satisfied by taking real u in the case of Type -1A solution. There is another solution for the reality requirement, which is by taking $u = w + iK'$, with real w (to be called as the Type-1B solution). In the Type-1B case, we obtain following expressions by substituting $u = w + iK'$ in the corresponding expressions of the Type-1A solution;

$$\partial\phi_{c-s}(z, \bar{z}) = \frac{2}{k}\sqrt{\frac{\beta}{V}}\mathrm{dn}(\chi, k) + 4k\sqrt{\frac{\beta}{V}}\frac{\mathrm{sn}(w, k)\ \mathrm{cn}(w, k)}{\mathrm{dn}(w, k)}\frac{S}{S^2 + 1}, \tag{13}$$

where

$$S = \frac{a\mathrm{cn}(\chi - w, k)e^{M}\Theta_s(\chi - w - iK')/\mathrm{sn}(\chi - w, k) - b\mathrm{dn}(w, k)e^{-M}\Theta_s(\chi + w + iK')}{b\mathrm{cn}(\chi + w, k)e^{-M}\Theta_s(\chi + w + iK')/\mathrm{sn}(\chi + w, k) + a\mathrm{dn}(w, k)e^{M}\Theta_s(\chi - w - iK')}, \tag{14}$$

and

$$M = -\sqrt{V\beta}\frac{\mathrm{dn}(w, k)^4 + k^2 - 1}{k^3\mathrm{sn}(w, k)\ \mathrm{cn}(w, k)\ \mathrm{dn}(w, k)}\bar{z} + \left(\frac{\Theta_s'(w + iK')}{\Theta_s(w + iK')} + \frac{\mathrm{cn}(w, k)(k^2\mathrm{sn}(w, k)^2 - 2)}{2\mathrm{dn}(w, k)\mathrm{sn}(w, k)}\right)\chi. \tag{15}$$

4.2 Properties of the solution

Figure 3(a) plots $\partial\phi_{c-s}$ of Type-1B as a function of time t and space x. The parameters used are those of Fig.1, i.e., $V = 1, k = 0.8, u = 0.7, \beta = 1, a = b = 1$. The distinct difference between the Type-1A and Type-1B solution can be seen from a time sliced view of the soliton at $t = 0$, see Fig. 3(b) and compare it with Fig. 1(b). The intrinsic velocity v_0 of the Type-1B soliton can be obtained from that of the Type-1A soliton by substituting $u \to w+iK'$, which gives $v_0(u = w+iK') = -v_0(w)$. Thus the characteristics of the intrinsic velocity v_0 of the Type-1B soliton is essentially the same as those of the Type-1A soliton. The shift of crests of the cnoidal wave can be similarly calculated as in the Type-1A solution, which is $4w$.

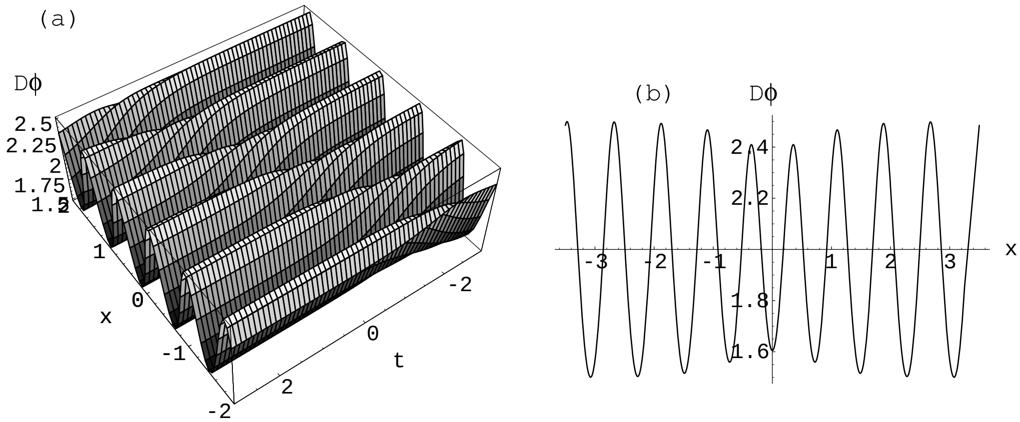

Figure 3: $\partial\phi_{c-s}$ showing a sine-Gordon soliton residing on a cnoidal wave background (Type-1B solution); (a) 3-dimensional plot, (b) time-sliced view at $t = 0$. The parameters are $V = 1, k = 0.8, u = 0.7, \beta = 1, a = b = 1$.

5 Type-2 soliton solution on a cn-type background

5.1 soliton solution

The "soliton + cnoidal wave" solution with the $\phi_c^{(2)}$ background can be obtained by substituting $k \to 1/k, u \to ku, \chi \to k^2\chi$ in the Type-1A solution. We call it the Type-2 solution. Note that $\mathrm{dn}(\chi, k) \to \mathrm{dn}(k^2\chi, 1/k) = \mathrm{cn}(k\chi, k)$ such that $\partial\phi_c^{(1)}(z,\bar{z}) \to \partial\phi_c^{(2)}(z,\bar{z})$. In this case

$$\partial\phi_{c-s}(z,\bar{z}) = 2k\sqrt{\frac{\beta}{V}}\mathrm{cn}(k\chi, k) + 4\sqrt{\frac{\beta}{V}}\frac{\mathrm{cn}(u,k)}{\mathrm{sn}(u,k)\ \mathrm{dn}(u,k)}\frac{S}{S^2+1}, \tag{16}$$

where

$$S = -\frac{ak\mathrm{sn}(u,k)\mathrm{cn}(k\chi - u, k)e^{M}\Theta_t(\chi_2 - u) + b\mathrm{dn}(u,k)e^{-M}\Theta_t(\chi_2 + u)}{bk\mathrm{sn}(u,k)\mathrm{cn}(k\chi + u, k)e^{-M}\Theta_t(\chi_2 + u) + a\mathrm{dn}(u,k)e^{M}\Theta_t(\chi_2 - u)}, \tag{17}$$

with

$$M = \sqrt{V\beta}\frac{1 - 2k^2\mathrm{sn}(u,k)^2 + k^2\mathrm{sn}(u,k)^4}{\mathrm{sn}(u,k)\ \mathrm{cn}(u,k)\ \mathrm{dn}(u,k)}\bar{z} + \left(k\frac{\Theta_t'(u)}{\Theta_t(u)} + \frac{k\mathrm{cn}(u,k)(1 - 2k^2\mathrm{sn}(u,k)^2)}{2\ \mathrm{dn}(u,k)\ \mathrm{sn}(u,k)}\right)\chi, \tag{18}$$

and

$$\Theta_t(u) = \theta_0\left(\frac{\pi u}{2(K - iK')}\right) = 1 + 2\sum(-)^n q^{n^2}\cos(\frac{n\pi u}{K - iK'}), \tag{19}$$

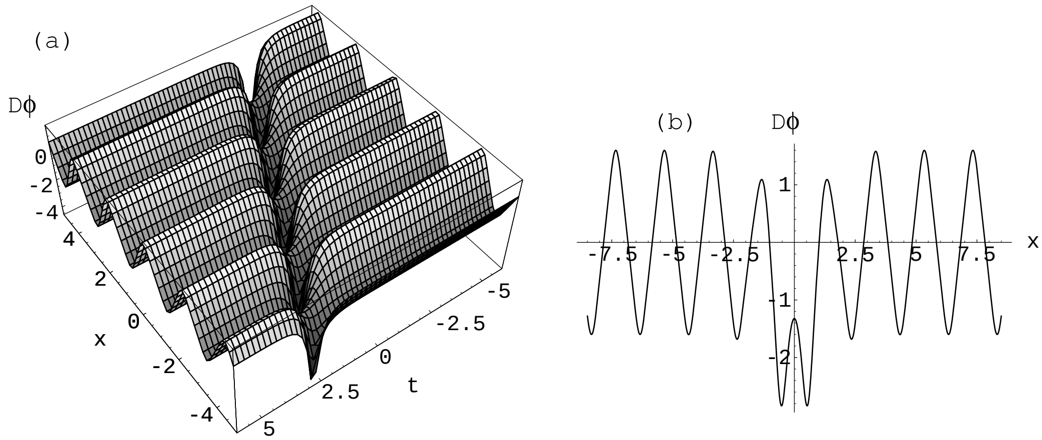

Figure 4: $\partial\phi_{c-s}$ showing a sine-Gordon soliton residing on a cnoidal wave background (Type-2 solution); (a) 3-dimensional plot, (b) time-sliced view at $t = 0$. The parameters are $V = 1, k = 0.8, u = 0.7, \beta = 1, a = b = 1$.

with $q = \exp[-\pi K'/(K - iK')]$.

5.2 Properties of the solution

Figure 4(a) plots $\partial\phi_{c-s}$ of the Type-2 soliton as a function of time t and space x. The parameters used are those of Fig.1, i.e., $V = 1, k = 0.8, u = 0.7, \beta = 1, a = b = 1$. The distinct difference between the Type-2 and Type-1 solutions can be seen from a time sliced view of the soliton at $t = 0$, see Fig. 4(b) and compare it with Fig. 1(b) and Fig. 2(b). The intrinsic velocity v_0 of the Type-2 soliton can be obtained from that of Type-1A soliton by substituting $k \to 1/k, u \to ku$, which gives

$$v_0 = \frac{\mathrm{dn}^2 u\ \mathrm{sn}^2 u + \mathrm{cn}^2 u\ (1 - 2k^2 \mathrm{sn}^2 u)}{\mathrm{sn} u\ \mathrm{cn} u\ \mathrm{dn} u(-4\Theta'_w(u)/\Theta_w(u) + 2\pi u/(KK')) + \mathrm{dn}^2 u\ \mathrm{sn}^2 u - \mathrm{cn}^2 u\ (1 - 2k^2 \mathrm{sn}^2 u)}, \tag{20}$$

where we use the identity

$$\frac{\Theta'_t(u)}{\Theta_t(u)} = -\pi \frac{u}{2(K - iK')K'} + \frac{\Theta'_w(u)}{\Theta_w(u)} \tag{21}$$

with

$$\Theta_w(u) = \theta_2(-i\frac{\pi u}{2K'}) = 2q^{1/4} \sum q^{n^2+n} \cos(\frac{-i(2n+1)\pi u}{2K'}), \tag{22}$$

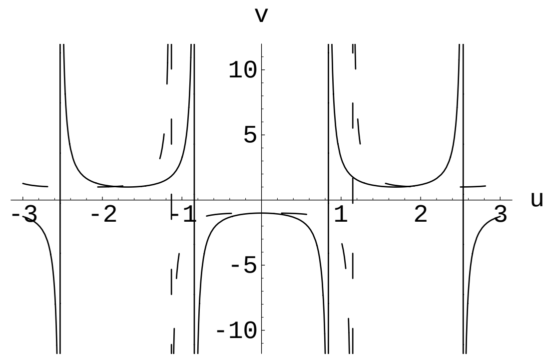

Figure 5: Intrinsic velocity v_0 versus u of the Type-2 sine-Gordon soliton for $k = 0.5$ (solid line) and $k = 0.9$ (dashed line). .

with $q = \exp(-\pi K/K')$. The characteristics of the intrinsic velocity v_0 of Type-2 soliton is shown in Fig. 5. Special values are $v_0 = -1$ at $u = 2nK$, n=integer and $v_0 = 1$ at $u = (2n+1)K$. At $u = (n+1/2)K$, $v_0 = \pm\infty$. It shows that type-2 solitons move with the velocity $|v| > 1$. We can obtain soliton solutions having $|v| < 1$, by taking the Bäckrund transformation $(\phi_{c-s}, x, t) \leftrightarrow (\phi_{c-s}+\pi, t, x)$. The shift of crests of the cnoidal wave can be similarly calculated as in the Type-1A solution, which gives $4u$ in the Type-2 solution. The substitution $u = w + iK'$ (used to obtain the Type-1B solution from the Type-1A solution) does not give new solution in the case of Type-2 solution. In fact, there exists a symmetry $u \leftrightarrow -u + K + iK'$ in the Type-2 solution.

In the $k \to 0$ limit of the Type-2 solution, $\phi_c^{(2)} \to 0$. Using the fact that $K(k=0) = \pi/2, K'(0) = \infty$, we obtain $S = -\exp(-2M + \ln b/a)$ where

$$M = \sqrt{V\beta}\left(\frac{\sin u}{\cos u} + \frac{\cos u}{\sin u}\right)\bar{z} + \frac{\cos u}{2\sin u}k\chi. \tag{23}$$

Especially at $u = K/2 = \pi/4$ with $a = b$ ($\beta = V = 1$), we obtain

$$\phi_{c-s} = -2\tan^{-1}\exp(4t),\ \partial\phi_{c-s} = -2\mathrm{sech}4t,\ \sin 2\phi_{c-s} = 2\mathrm{sech}4t\ \tanh 4t. \tag{24}$$

Considering the $x \leftrightarrow t, \beta \leftrightarrow -\beta$ symmetry of the sine-Gordon theory, the solution in Eq. (24) becomes the well-known sech-type soliton (without a background) of $\beta = -1$ theory.

6 Discussion

In this paper, we introduce "soliton+cnoidal wave" solutions of the sine-Gordon equation. It was obtained using the DT and the Sym's solution of the associated linear problem on a cnoidal wave background. The solutions are expressed in terms of the Jacobi elliptic functions and can be easily manipulated to obtain interesting physical characteristics. In fact, we use the symbolic package MATHEMATICA to check various formulae of the present paper including the solutions itself, as well as to obtain the figures. We calculate the velocity of a soliton on a cnoidal wave and the shift of crests of the cnoidal wave. The velocity of the soliton is given by the relativistic addition of two velocities, the velocity of the cnoidal wave and the intrinsic velocity of the soliton observed in the moving frame of the cnoidal wave. The intrinsic velocity is determined by the property of the cnoidal wave (dn or cn wave and the modulus k) and the DT parameter u. It would be interesting to make an (numerical) analysis about the nature of the solitons, especially when it lies on a finite-width cnoidal wave background.

The stability analysis of these solutions is remained for future study. There already appeared that linearized instabilities of N-phase solutions can be labeled in terms of spectral data [17]. It implies that there exist regions in the parameter space of u, k and L (finite length of cnoidal waves), where the solutions are stable. Explicit relation between the spectral data and DT parameter u is not understood yet.

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